

ON THE NUMBER OF p' -DEGREE CHARACTERS IN A FINITE GROUP

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ABSTRACT. Let p be a prime divisor of the order of a finite group G . Then G has at least $2\sqrt{p-1}$ complex irreducible characters of degrees prime to p . In case p is a prime with $\sqrt{p-1}$ an integer this bound is sharp for infinitely many groups G .

1. INTRODUCTION

Let p be a prime and G a finite group. Denote the set of complex irreducible characters of G whose degrees are prime to p by $\text{Irr}_{p'}(G)$. The McKay Conjecture states that $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(P))|$ where $N_G(P)$ is the normalizer of a Sylow p -subgroup P in G . Some known cases (easy consequence of [5, Thm. 1] and a special case of [7]) of this problem together with a recent result of the second author [11] stating that the number of conjugacy classes in a finite group G is at least $2\sqrt{p-1}$ whenever p is a prime divisor of the order of G allows us to prove the following.

Theorem 1.1. *Let G be a finite group and p a prime divisor of the order of G . Then $|\text{Irr}_{p'}(G)| \geq 2\sqrt{p-1}$.*

Our proof of Theorem 1.1 shows that $|\text{Irr}_{p'}(G)|$ is smallest possible for a finite group G whose order is divisible by a prime p if and only if the normalizer of a Sylow p -subgroup of G has a certain special structure. This may be natural in view of the (unsolved) McKay Conjecture. Our second theorem gives a complete description of finite groups G with the property that $|\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$ for a prime divisor p of the order of G , consistent with the McKay conjecture.

Theorem 1.2. *Let G be a finite group, p a prime divisor of the order of G , and P a Sylow p -subgroup of G . Suppose that $\sqrt{p-1}$ is an integer and set H to be the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$ (whose subgroup of order p is self centralizing). Then $|\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$ if and only if $N_G(P) \cong H$.*

Moreover this happens if and only if $G \cong H$, or $O_{p'}(G) = F(G)$, the subgroup $F(G)P$ is a Frobenius group, and $G/F(G)$ is either isomorphic to H or is an almost simple group A as described below.

- (1) $p = 5$ and $A = \mathfrak{A}_5, \mathfrak{A}_6, \text{L}_2(11)$ or $\text{L}_3(4)$;
- (2) $p = 17$ and $A = \text{S}_4(4), \text{O}_8^-(2)$ or $\text{L}_2(16).2$;

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- (3) $p = 37$ and $A = {}^2G_2(27)$ or $U_3(11).2$;
- (4) $p = 257$ and $A = S_{16}(2)$, $O_{18}^-(2)$, $L_2(256).8$, $S_4(16).4$, $S_8(4).2$, $O_8^-(4).4$, $O_{16}^-(2).2$ or $F_4(4).2$.

In Proposition 6.3 we show that for any prime p with $\sqrt{p-1}$ an integer there are in fact infinitely many finite solvable groups G with $|\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$. We remark that it is an open problem first posed by Landau whether there are infinitely many primes p with $\sqrt{p-1}$ an integer (see e.g. [13, Sec. 19]).

2. THE MCKAY CONJECTURE

Let G be a finite group and p a prime. The McKay Conjecture claims that $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(P))|$ where $N_G(P)$ is the normalizer of a Sylow p -subgroup P in G . Thus if we wish to bound $|\text{Irr}_{p'}(G)|$ and assume the validity of the McKay Conjecture for G and p , then we may assume that the Sylow p -subgroup P is normal in G . In this case we have $|\text{Irr}_{p'}(G)| \geq |\text{Irr}_{p'}(G/\Phi(P))|$ where $\Phi(P)$ is the Frattini subgroup in P , a normal subgroup of G . Since $P/\Phi(P)$ is an elementary abelian normal subgroup in $G/\Phi(P)$ which is also the Sylow p -subgroup of $G/\Phi(P)$, by Clifford theory we have that all complex irreducible characters of $G/\Phi(P)$ have degrees prime to p . But the number of conjugacy classes of $G/\Phi(P)$ is at least $2\sqrt{p-1}$ by [11, Thm. 1.1] with equality if and only if $\sqrt{p-1}$ is an integer and $G/\Phi(P)$ is the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$ (whose subgroup of order p is self centralizing).

Now let us suppose that the McKay Conjecture is true for a finite group G and a prime p . Then $|\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$ if and only if the same holds in case G contains a normal Sylow p -subgroup P . By the previous paragraph, $|P/\Phi(P)| = p$ so P is cyclic. But then, by Clifford theory once again, all complex irreducible characters of G have degrees prime to p . Finally, by [11, Thm. 1.1], the number of conjugacy classes of G is equal to $2\sqrt{p-1}$ if and only if G is the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$.

By the previous two paragraphs we showed Theorem 1.1 and the first half of Theorem 1.2 in case the McKay Conjecture is true for the pair G and p . The McKay Conjecture is known to be true, for example, for groups with a cyclic Sylow p -subgroup, by Dade [5, Thm. 1].

3. REDUCTION

In this section we prove a reduction of Theorem 1.1 and of the first half of Theorem 1.2 to a question on finite non-abelian simple groups.

Let G be a finite group and p a prime dividing the order of G . By the previous section we can assume that the Sylow p -subgroups of G are not cyclic. So we would like to show $|\text{Irr}_{p'}(G)| > 2\sqrt{p-1}$ in all remaining cases.

From the well-known identity $|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2$ we see that $|\text{Irr}_{p'}(G)| > 2\sqrt{p-1}$ is true for $p = 2$ and $p = 3$. So assume from now on that $p \geq 5$.

3.1. Reduction to the monolithic case. Let G be a minimal counterexample to the bound, that is, $|\text{Irr}_{p'}(G)| \leq 2\sqrt{p-1}$ and G does not have a cyclic Sylow p -subgroup.

Let N be a minimal normal subgroup in G . Suppose first that $|G/N|$ is divisible by p . Then $|\text{Irr}_{p'}(G)| \geq |\text{Irr}_{p'}(G/N)| \geq 2\sqrt{p-1}$ by the minimality of G . So both inequalities

must be equalities. But then G/N has a Sylow p -subgroup of order p and p^2 divides

$$\sum_{\chi \in \text{Irr}(G) \setminus \text{Irr}(G/N)} \chi(1)^2 = |G| - |G/N|.$$

This implies that p^2 cannot divide $|G|$ (only p). But we excluded the case when G has a cyclic Sylow p -subgroup.

So we must have that $|G/N|$ is not divisible by p , whence $|N|$ is divisible by p . Then N is an elementary abelian p -group or is a direct product of simple groups S having order divisible by p . By this argument it also follows that N is the unique minimal normal subgroup of G . If N is abelian then $\text{Irr}_{p'}(G) = \text{Irr}(G)$ by Clifford theory and so we get the result by [11, Thm. 1.1].

Thus $N = S_1 \times \cdots \times S_t$ where all S_i 's are isomorphic to a non-abelian simple group S having order divisible by p . Note that G/N permutes the simple factors transitively (but not necessarily faithfully).

3.2. Reduction to simple groups. We continue the investigation of a minimal counterexample G as in the previous subsection. If $\psi \in \text{Irr}_{p'}(N)$ then any irreducible character of G lying above ψ has p' -degree by Clifford theory.

We wish to give a lower bound for the number of G/N -orbits on the set $\text{Irr}_{p'}(N)$. For this we may assume that G/N is as large as possible, subject to our conditions. So we may assume that $G = A \wr T$ where $\text{Inn}(S) \leq A \leq \text{Aut}(S)$ is a group for which $|A/\text{Inn}(S)|$ is prime to p and T is a transitive permutation group on t letters with $|T|$ coprime to p (but we may and will take T to be \mathfrak{S}_t). Let A_1 be the stabilizer of S_1 in G . Let K_1 be the normal subgroup of A_1 consisting of those elements which induce inner automorphisms on S_1 . Then A_1/K_1 can be considered as a p' -subgroup of $\text{Out}(S_1)$. Let k be the number of A_1 -orbits on $\text{Irr}_{p'}(S_1)$. Then $|\text{Irr}_{p'}(G)| \geq \binom{k+t-1}{t}$.

Suppose for a moment that $t \geq 2$. Then $|\text{Irr}_{p'}(G)| \geq \binom{k+1}{2} = k(k+1)/2$. We want this to be larger than $2\sqrt{p-1}$. This is certainly true if $k \geq 2(p-1)^{1/4}$. On the other hand for $t = 1$ we have $G = A$ and so we need $|\text{Irr}_{p'}(G)| > 2\sqrt{p-1}$.

Thus Theorem 1.1 and the first part of Theorem 1.2 is a consequence of the following result.

Theorem 3.1. *Let S be a finite non-abelian simple group whose order is divisible by a prime p at least 5. Suppose that S is not isomorphic to a projective special linear group $L_2(q)$, a Suzuki group ${}^2B_2(q^2)$ or a Ree group ${}^2G_2(q^2)$. Let $X \leq \text{Aut}(S)$ be a group containing $\text{Inn}(S)$ so that $|X/\text{Inn}(S)|$ is not divisible by p . Furthermore let k be the number of X -orbits on $\text{Irr}_{p'}(S)$. Then*

- (a) $k \geq 2(p-1)^{1/4}$; and
- (b) *if the Sylow p -subgroups of X are not cyclic then $|\text{Irr}_{p'}(X)| > 2\sqrt{p-1}$.*

Note that we may exclude the rank 1 groups $L_2(q)$, ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$ in Theorem 3.1. Indeed, by Theorems A and B and by the comments in between on page 35 of [7], we see that the McKay Conjecture is true for any corresponding G . So we may as well assume that S is different from these groups.

Note that if X is as in Theorem 3.1 then it is sufficient (but not necessary) to show that $|\text{Irr}_{p'}(X)| > 2\sqrt{p-1} \cdot |X/S|$.

4. ALTERNATING AND SPORADIC SIMPLE GROUPS

The aim of this section is to prove Theorem 3.1 for alternating and sporadic groups.

4.1. The case when $S = \mathfrak{A}_n$. Let us exclude the case $n = 6$ from the discussion below because in this case the full automorphism group of S is not \mathfrak{S}_n .

We begin with a result of Macdonald (the following form of which can be found in a paper by Olsson [12]). For a non-negative integer m let $\pi(m)$ denote the number of partitions of m . An m -split of a non-negative integer s is a sequence of non-negative integers (s_1, \dots, s_m) so that $\sum_{i=1}^m s_i = s$. Put $k(m, s) = \sum \pi(s_1)\pi(s_2) \cdots \pi(s_m)$ where the sum is over all m -splits of s . (Notice that $k(m, 0) = 1$.) For a prime divisor p of $|\mathfrak{S}_n|$ let the p -adic expansion of the integer n be $a_0 + a_1p + \cdots + a_rp^r$. Then Macdonald's result states that

$$|\text{Irr}_{p'}(\mathfrak{S}_n)| = k(1, a_0)k(p, a_1) \cdots k(p^r, a_r).$$

Notice that $m \cdot s \leq k(m, s)$ for all m and s . This gives $p - 1 \leq n - 1 \leq |\text{Irr}_{p'}(\mathfrak{S}_n)|$ since the product of integers each at least 2 is always at least their sum. Thus

$$|\text{Irr}_{p'}(\mathfrak{A}_n)| \geq k \geq (n - 1)/2 \geq (p - 1)/2.$$

A simple calculation shows that this is larger than $2\sqrt{p-1}$ unless $p \leq 17$. So we may assume that $5 \leq p \leq 17$, otherwise we are done. But the same calculation can be applied using n in place of p . So we may also assume that $n \leq 17$.

If $a_0 \geq 3$ or if $a_1 \geq 2$ or if $a_i \geq 1$ for some $i \geq 2$, then $|\text{Irr}_{p'}(\mathfrak{S}_n)| \geq 3p$. Using this bound and the calculation referred to in the previous paragraph we get an affirmative answer to the problem. So only the following cases are to be considered.

- (1) $n = p = 5, 7, 11, 13, 17$. In this case $|\text{Irr}_{p'}(\mathfrak{S}_n)| = p$.
- (2) $n = p + 1 = 8, 12, 14$. In this case $|\text{Irr}_{p'}(\mathfrak{S}_n)| = p$.
- (3) $n = p + 2 = 7, 9, 13, 15$. In this case $|\text{Irr}_{p'}(\mathfrak{S}_n)| = 2p$.

For all the above values of n and p still to be considered (even for $n = 6$) we have that a Sylow p -subgroup of X has order p , that is, is cyclic. So we only have to bound k .

In the exceptional cases (1)–(3) above we certainly have $k \geq (p + 1)/2$ since p is odd. But then the bound in (a) of Theorem 3.1 holds for $p \geq 5$.

Now suppose that $n = 6$. It is sufficient to show in this case that $k \geq 2(p - 1)^{1/4}$ (where p here is 5). Since the complex irreducible character degrees of \mathfrak{A}_6 are 1, 5, 5, 8, 8, 9, 10, we certainly have $k \geq 3$. But 3 is larger than our proposed bound.

4.2. The case when S is sporadic. For sporadic groups and ${}^2F_4(2)'$ it is straightforward to check the validity of the conditions in Theorem 3.1 from the known character tables in [4].

5. GROUPS OF LIE TYPE

Here, we prove Theorem 3.1 for groups of Lie type. Let $G = \mathbf{G}^F$ be the group of fixed points under a Steinberg endomorphism F of a simple algebraic group \mathbf{G} of adjoint type over an algebraically closed field of characteristic r . Let p be a prime (which may coincide with r) dividing $|G|$. Let S be the simple socle of G .

5.1. Two easy observations. As above, G is a finite reductive group of adjoint type.

Lemma 5.1. *Suppose that p does not divide $|G/S|$. Then the claim of Theorem 3.1 holds for (S, p) if $2\sqrt{p-1} \cdot |\text{Out}(S)|_{p'} < |\text{Irr}_{p'}(G)|$.*

Proof. By the condition on G , by Schreier's conjecture, and by Hall's theorem, we may assume that X contains G . Now $2\sqrt{p-1} \cdot |\text{Out}(S)|_{p'} < |\text{Irr}_{p'}(G)|$ implies that $2\sqrt{p-1} \cdot |X/S| < |\text{Irr}_{p'}(G)|$. From this we have

$$2\sqrt{p-1} < \frac{|G|}{|X|} \cdot \frac{|\text{Irr}_{p'}(G)|}{|G:S|} \leq \frac{|G|}{|X|} \cdot \left(\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| \right) \leq \frac{1}{|X|} \sum_{g \in X} |\text{fix}(g)| = k$$

where $|\text{fix}(g)|$ denotes the number of fixed points of $g \in X$ on $\text{Irr}_{p'}(S)$. \square

Here is a further easy sufficient criterion:

Lemma 5.2. *Let S be non-abelian simple. Assume that there is $I \subseteq \text{Irr}_{p'}(S)$ such that all $\chi \in I$ are $\text{Out}(S)$ -invariant and extend to $\text{Aut}(S)$. Then the conclusion of Theorem 3.1 holds for (S, p) if one of the following conditions holds:*

- (1) $p < |I|^2/4 + 1$, or
- (2) Sylow p -subgroups of $\text{Aut}(S)$ are cyclic and $p \leq |I|^4/16 + 1$.

Proof. By assumption $\text{Out}(S)$ has at least $k := |I|$ orbits on $\text{Irr}_{p'}(S)$. Since all characters of I extend to $\text{Aut}(S)$, any $S \leq X \leq \text{Aut}(S)$ has $|\text{Irr}_{p'}(X)| \geq k$. Now $k = |I| > 2(p-1)^{1/2} \geq 2(p-1)^{1/4}$, so (S, p) satisfies the condition in Theorem 3.1(b). If Sylow p -subgroups of $\text{Aut}(S)$ are cyclic, we just need $k > 2(p-1)^{1/4}$. \square

Note that for invariant characters extendibility to $\text{Aut}(S)$ is automatically satisfied if all Sylow subgroups of $\text{Out}(S)$ are cyclic, for example.

5.2. The defining characteristic case (for rank $l \geq 2$).

Proposition 5.3. *Theorem 3.1 holds for S of Lie type in characteristic p .*

Proof. As before, let \mathbf{G} be a simple linear algebraic group in characteristic p of adjoint type with a Steinberg endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ and $G := \mathbf{G}^F$ such that $S = [G, G]$. All finite simple groups of Lie type are of this form (see [10, Prop. 24.21]). We denote by (\mathbf{G}^*, F^*) the dual pair of (\mathbf{G}, F) (see [3, Sec. 4.2]). Here \mathbf{G}^* is a simple algebraic group of simply connected type. We denote the corresponding finite group of Lie type by G^* . By [10, Prop. 24.21], we have $G^*/Z(G^*) \cong [G, G] = S$. Since $p \geq 5$, we know by [2, Lemma 5] that the set of p' -degree complex irreducible characters of G is precisely the set of semisimple characters of G , whose elements are labelled by representatives of the conjugacy classes of semisimple elements of G^* . Thus $|\text{Irr}_{p'}(G)| = q^l$ where l is the semisimple rank of \mathbf{G}^* , and q is the absolute value of all eigenvalues of F on the character group of an F -stable maximal torus of \mathbf{G} , by [3, Thm. 3.7.6(ii)].

By Clifford theory we then have

$$q^l = |\text{Irr}_{p'}(G)| \leq |G:S| \cdot t$$

where t is the number of G/S -orbits on $\text{Irr}_{p'}(S)$. By the orbit-counting lemma,

$$q^l \leq |G:S| \cdot t = \sum_{g \in G/S} |\text{fix}(g)| \leq \sum_{g \in \text{Out}(S)} |\text{fix}(g)| \leq k \cdot |\text{Out}(S)|.$$

So we get $q^l/|\text{Out}(S)| \leq k$.

In order to prove Theorem 3.1 for (S, p) it is sufficient to see that $q^l/|\text{Out}(S)| > 2\sqrt{p-1}$, where $q = p^f$. Bounds for $|\text{Out}(S)|$ can be read off from [4, Tab. 5]. If $(f, l, p) \neq (1, 2, 5)$ nor $(1, 2, 7)$, then the bound $|\text{Out}(S)| \leq (6l+3)f$ is sufficient for our purposes (note that $l \geq 2$). On the other hand, if $(f, l, p) = (1, 2, 5)$ or $(1, 2, 7)$ then the bounds $|\text{Out}(S)| \leq 6$ and $|\text{Out}(S)| \leq 8$ are sufficient, respectively. \square

5.3. Exceptional type groups in non-defining characteristic.

Proposition 5.4. *Let S be a simple exceptional group of Lie type, not of type 2B_2 or 2G_2 , and $p \geq 5$ a prime dividing $|S|$ but different from the defining characteristic. Then (S, p) satisfies the conclusion of Theorem 3.1.*

Proof. Let G be a finite reductive group of adjoint type with socle S . We first deal with the primes p for which Sylow p -subgroups of G are non-abelian. These necessarily divide the order of the Weyl group W of G , so $p \leq 7$, and G is of type ${}^{(2)}E_6$, E_7 or E_8 . Furthermore, $p|(q \pm 1)$ if $p = 7$, or if $p = 5$ and G is not of type E_8 . It is then straightforward to check (for example from the tables in [3, §13.9]) that G has at least as many unipotent characters of p' -degree as given in Table 1. Since unipotent characters extend to $\text{Aut}(S)$ by [9, Thm. 2.5], the claim follows from Lemma 5.2 in this case.

TABLE 1. Invariant unipotent characters, $p \in \{5, 7\}$

G	${}^{(2)}E_6$	E_7	E_8
$p = 5$	10	30	20
$p = 7$	—	14	28

We may now assume that Sylow p -subgroups of G are abelian. Then there exists a unique cyclotomic polynomial Φ_d dividing the generic order of G and such that $p|\Phi_d(q)$. Moreover, there exists a maximal torus T_d of G containing a Sylow d -torus of G , and so in particular a Sylow p -subgroup of G (see [10, Thm. 25.14]). Let $\Phi_d^{a_d}$ be the precise power of Φ_d dividing the order polynomial of G . The Sylow p -subgroups of G are cyclic if and only if $a_d = 1$. Let W_d be the relative Weyl group of T_d . Then by generalized Harish-Chandra theory (or alternatively from the formulas in [3, §13.9]) there exist at least $|\text{Irr}(W_d)|$ many unipotent characters of G of p' -degree. By [9, Thms. 2.4 and 2.5] all of these extend to $\text{Aut}(S)$ unless G is of type G_2 and $r = 3$, or of type F_4 and $r = 2$. The various W_d and a_d are explicitly known (see e.g. [1, Tables 1 and 3]), and applying Lemma 5.2 we conclude that our claim holds if p is as in Table 2. Here, the left-most half of the table contains the cases with $a_d > 1$, while in the right-most part we have $a_d = 1$, so Sylow p -subgroups are cyclic.

So from now on we suppose that p is larger than the bound given in the table. Let d, T_d, W_d be as above. Let $s \in T_d$ be semisimple. Then s centralizes a Sylow p -subgroup of G , so the semisimple character in the Lusztig series $\mathcal{E}(G, s)$ has degree prime to p by Lusztig's Jordan decomposition (see e.g. [8, Prop. 7.2]). Since fusion of semisimple elements in maximal tori is controlled by the relative Weyl group, there exist at least

TABLE 2. $\text{Aut}(S)$ -invariant unipotent characters

G	d	#	p	d	#	p
G_2	1, 2	6	$p \leq 10$	3, 6	6	$p \leq 82$
3D_4	1, 2	6	$p \leq 10$	12	4	$p \leq 17$
	3, 6	7	$p \leq 13$			
2F_4	1, 4, 8', 8''	7	$p \leq 13$	12, 24', 24''	12	$p \leq 1297$
F_4	1, 2	11	$p \leq 31$	8, 12	≥ 8	$p \leq 257$
	3, 6	9	$p \leq 21$			
${}^{(2)}E_6$	1, 2, 3, 4, 6	≥ 16	$p \leq 65$	5, 8, 9, 12, (10, 18)	≥ 5	$p \leq 40$
E_7	1, 2, 3, 4, 6	≥ 48	$p \leq 577$	5, 7, 8, 9, 10, 12, 14, 18	≥ 14	$p \leq 2402$
E_8	1, 2, 3, 4, 6	≥ 59	$p \leq 871$	7, 9, 14, 18	≥ 28	$p \leq 38417$
	5, 8, 10, 12	≥ 32	$p \leq 257$	15, 20, 24, 30	≥ 20	$p \leq 10001$

$|T_d|/|W_d|$ semisimple conjugacy classes of G with representatives in T_d , whence $|\text{Irr}_{p'}(G)| \geq |T_d|/|W_d|$. We now go through the various types of groups.

Let first $G = S = G_2(q)$ with $q = r^f > 2$ (as $G_2(2) \cong \text{Aut}(U_3(3))$). Then $\text{Out}(S)$ is cyclic of order f for $r \neq 3$ respectively $2f$ for $r = 3$, and $d \in \{1, 2, 3, 6\}$, with $a_d = 2$ for $d = 1, 2$ and $a_d = 1$ else. Table 2 then shows that $q \geq 11$. It is now straightforward to check that $|T_d|/|W_d| > 2\sqrt{p-1}|\text{Out}(S)|$, so the condition in Lemma 5.1 is satisfied in these cases.

Next consider $G = S = {}^3D_4(q)$, $q = r^f$. As before, $\text{Out}(S)$ is cyclic, of order $3f$. Here, we have $d \in \{1, 2, 3, 6, 12\}$, with $a_d = 2$ for $d \leq 6$. By Table 2 we may assume that $q \geq 11$. In all cases the estimate above gives the claim. The same arguments also apply to ${}^2F_4(2^{2f+1})$ and $F_4(q)$.

Now assume that $G = S = E_6(q)$, $q = r^f$. Here the outer automorphism group is of order $2f \gcd(3, q-1)$, but no longer cyclic. We have $d \in \{1, 2, 3, 4, 5, 6, 8, 9, 12\}$. First assume that Sylow p -subgroups are cyclic, so $d \in \{5, 8, 9, 12\}$. Then $p \geq 41$ by Table 2, and $|W_d| \leq 12$. The standard estimate now applies. For $d \in \{2, 3, 4, 6\}$ we have $67 \leq p \leq q^2 + 1$, while $|T_d| \geq (q^2 - q)^3$ and $|W_d| \leq 1152$, while for $d = 1$ we have $67 \leq p \leq q - 1$ and $|T_d| = (q - 1)^6$. In all cases we obtain a contradiction to the standard estimate. The case of ${}^2E_6(q)$ can be handled similarly. For $E_7(q)$ the outer automorphism group has order $f \gcd(2, q-1)$, and the same approach as before applies. Finally, let $G = S = E_8(q)$ with $q = r^f$. Then $|\text{Out}(S)| = f$. We now discuss the various possibilities for d . If $d = 1$, so $p|(q-1)$, then W_d is the Weyl group of G , with $|\text{Irr}(W_d)| = 112$. So we are done whenever $2f\sqrt{p-1} < 112$, which certainly is the case for $q \leq 1000$. For $q \geq 1001$ we have

$$\Phi_d(q)^a/|W_d| = (q-1)^8/696729600 > 2\log_p(q)\sqrt{p-1}.$$

The case $d = 2$ is very similar. For $d = 3$ or $d = 6$, $|W_d| = 155520$ (see [1, Table 3]) and $|\text{Irr}(W_d)| = 102$. We may conclude as before. Similarly, for $d = 4$ we have $|W_d| = 46080$ and $|\text{Irr}(W_d)| = 59$; for $d = 5$ or $d = 10$ we have $|W_d| = 600$ and $|\text{Irr}(W_d)| = 45$; for $d = 12$ we have $|W_d| = 288$ and $|\text{Irr}(W_d)| = 48$. Finally, for the cases $d \in \{7, 14, 9, 18, 15, 20, 24, 30\}$ with cyclic Sylow p -subgroups the estimates are even easier, using the bounds in Table 2. This achieves the proof. \square

5.4. Groups of classical type in non-defining characteristic.

Proposition 5.5. *Let S be a simple classical group of Lie type and $p \geq 5$ a prime dividing $|S|$ but different from the defining characteristic. Then (S, p) satisfies the conclusion of Theorem 3.1.*

Proof. Let first $G = \mathrm{SO}_{2n+1}(q)$ or $\mathrm{PCSp}_{2n}(q)$ with $q = r^f$ and $n \geq 2$. Here $\mathrm{Out}(S)$ is cyclic of order $f \gcd(2, q-1)$, respectively of order $2f$ if $n = 2$ and q is even. Let d be minimal such that p divides $q^d \pm 1$. A Sylow d -torus T_d of G has order Φ_d^a when $n = ad + s$ with $0 \leq s < d$. The centralizer of T_d in G has a subgroup of the form $(q^d \pm 1)^a G_s(q)$, where G_s has the same type as G and rank s (see [1, §3A]). The relative Weyl group W_d of T_d is the wreath product $C_{2d} \wr \mathfrak{S}_a$.

If Sylow p -subgroups of G are non-abelian, then $p \leq n$ divides $|W_d|$, whence $p \leq a$ as p cannot divide d . Now the number of unipotent characters of p' -degree of G in the principal p -block is at least the number of p' -characters of W_d , hence of its factor group \mathfrak{S}_a , hence at least $p-1$, and all of these are $\mathrm{Out}(S)$ -invariant by [9, Thm. 2.5], so we are done in this case.

Else, the centralizer of T_d contains a Sylow p -subgroup of G , whence all semisimple elements of the torus of order $(q^d \pm 1)^a$ give rise to semisimple characters of G in $\mathrm{Irr}_{p'}(G)$, and in addition the unipotent characters in the principal p -block of G , of which there are $|\mathrm{Irr}(W_d)|$ many, have degree coprime to p . Thus by Lemma 5.1 it suffices to show that

$$|\mathrm{Irr}(W_d)| + \frac{(q^d - 1)^a}{(2d)^a a!} > 2f \gcd(2, q-1) \sqrt{p-1}$$

where $p|(q^d \pm 1)$. If $a = 1$ then Sylow p -subgroups of $\mathrm{Aut}(G)$ are cyclic. Otherwise it is easily seen that this inequality always holds.

Next let $G = \mathrm{PCO}_{2n}^\pm(q)$ with $q = r^f$ and $n \geq 4$. Here $\mathrm{Out}(S)$ has order $fg \gcd(4, q^n \pm 1)$, where $g = 6$ for $n = 4$ and $g = 2$ else denotes the number of graph automorphisms. Let again d be minimal such that p divides $q^d \pm 1$. The situation is very similar to the one for groups of types B_n and C_n , except that the relative Weyl group W_d sometimes is a subgroup of index two in the wreath product $C_{2d} \wr \mathfrak{S}_a$. Arguing as before we find that there are no cases with $a > 1$ violating the above inequality. For $a = 1$ Sylow p -subgroups of G are cyclic.

Next let $G = \mathrm{PGL}_n(q)$ with $q = r^f$ and $n \geq 3$. Let d be minimal with p dividing $q^d - 1$ and write $n = ad + s$ with $0 \leq s < d$. A Sylow d -torus T_d of G has order Φ_d^a . The centralizer of T_d in G contains a subgroup of the form $(q^d - 1)^a G_s(q)$, where G_s is of type A_{s-1} . The relative Weyl group W_d of T_d is the wreath product $C_d \wr \mathfrak{S}_a$.

If Sylow p -subgroups of G are non-abelian, then $p \leq n$ divides $|W_d|$, and so $p \leq a$. Again, the number of unipotent characters of p' -degree of G in the principal p -block is at least the number of p' -characters of W_d , hence of \mathfrak{S}_a , hence at least $p-1$. Since all of these are $\mathrm{Out}(S)$ -invariant, we are done in this case.

Otherwise we may assume that $a > 1$. Arguing as in the case of the other classical groups, we arrive at the following inequality

$$|\mathrm{Irr}(W_d)| + \frac{(q^d - 1)^a}{d^a a!} > 2f \gcd(n, q-1) \sqrt{p-1},$$

which turns out to be satisfied for all relevant values.

The case of $G = \text{PGU}_n(q)$ is entirely similar, which $q^d - 1$ replaced by $q^d - (-1)^d$ throughout. The proof is complete. \square

6. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2.

Lemma 6.1. *Let G be a finite group, p a prime divisor of the order of G , and P a Sylow p -subgroup of G . Suppose that $\sqrt{p-1}$ is an integer and set H to be the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$ (whose subgroup of order p is self centralizing). Then $|\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$ if and only if $N_G(P) \cong H$. Moreover this happens if and only if $G \cong H$, or $O_{p'}(G) = F(G)$, the subgroup $F(G)P$ is a Frobenius group, and $G/F(G)$ is either isomorphic to H or is an almost simple group A with $N_A(F(G)P/F(G)) \cong H$.*

Proof. We have already proved the first statement of the lemma in the preceding sections.

So now suppose that $N_G(P) \cong H$ holds. Then by Theorem 1.1, we have

$$2\sqrt{p-1} \leq |\text{Irr}_{p'}(G/O_{p'}(G))| \leq |\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$$

and so $N_{G/O_{p'}(G)}(Q) \cong H$ for a Sylow p -subgroup Q of $G/O_{p'}(G)$. Since $O_{p'}(G/O_{p'}(G)) = 1$ and $|Q| = p$, we see that either Q is normal in $G/O_{p'}(G)$ and thus $G/O_{p'}(G) \cong H$, or $G/O_{p'}(G)$ is almost simple. Since P is self centralizing in G , it acts fixed point freely on $O_{p'}(G)$ and so $O_{p'}(G)P$ is a Frobenius group. By Thompson's theorem [14, Thm. 5.1'], $O_{p'}(G) \leq F(G)$. The other containment follows from $P \not\leq F(G)$ whenever $G \not\cong H$.

Now consider the other implication of the second statement of the lemma. Assume that $G \not\cong H$. Since $F(G)P$ is a Frobenius group, we have $N_G(P) \cap F(G) = 1$. Furthermore $N_{G/F(G)}(P) \cong N_{G/F(G)}(F(G)P/F(G)) \cong H$. \square

To finish the proof of Theorem 1.2, we need to classify almost simple groups A with the property that the normalizer of a Sylow p -subgroup in A is the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$ (whose subgroup of order p is self centralizing).

Proposition 6.2. *Let A be a finite almost simple group and p a prime. Then the Sylow p -subgroups of A are as described in Lemma 6.1 if and only if A is as in (1)–(4) of Theorem 1.2.*

Proof. Note that the smallest primes $p > 2$ such that $\sqrt{p-1}$ is an integer are given by 5, 17, 37, 101, 197, 257, ... Assume that A is a non-abelian almost simple group with socle S and with a Sylow p -subgroup as in Theorem 1.2. For S a sporadic group, it is readily checked from the Atlas [4] that no example arises (only the primes $p = 5, 17, 37$ are relevant). Now let $S = \mathfrak{A}_n$ with $n \geq 5$. Any element of \mathfrak{S}_n is rational, so any element of order p of \mathfrak{A}_n is conjugate to at least $(p-1)/2$ of its powers. But $(p-1)/2 \leq \sqrt{p-1}$ if and only if $p = 5$, and 5-cycles are non-rational only in \mathfrak{A}_5 and in \mathfrak{A}_6 . This occurs in exception (1).

If S is of Lie type in defining characteristic, its Sylow p -subgroups have order p only when $S = \text{L}_2(p)$, in which case the automizer has order $(p-1)/\gcd(p-1, 2)$. Again, only $p = 5$ and $A = \text{L}_2(5) = \mathfrak{A}_5$ arises.

Now assume that S is of Lie type but p is not the defining characteristic. Note that if p divides $|A|$, then it divides $|S|$, unless A contains a coprime field automorphism. But the latter have non-trivial centralizer in S , so indeed we may suppose that p divides

$|S|$. If p divides the order of the Weyl group of S , then p^2 divides $|S|$, so this is not the case. Otherwise Sylow p -subgroups of S are abelian and contained in some maximal torus T of S . In particular this torus must be of prime order p and self-centralizing. Let $m := |N_A(T)/T|$, then moreover $m^2 + 1 = |T| = p$. So in particular m has to be even. First assume that S is of exceptional Lie type. It is easily seen that under the above restrictions the only example is ${}^2G_2(27)$ with $p = 37$ as in (3), or $F_4(4).2$ with $p = 257$ as in (4). For example, for $A = E_8(q)$, $q = r^f$, the only possible values for m are $m = 15u, 20u, 24u, 30u$ where $u|f$, while $|T| \geq q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$ for cyclic maximal tori, which clearly gives no example.

Finally we handle the case that A is of classical Lie type. If A is of type $B_n(q)$ or $C_n(q)$ with $n \geq 2$ the only cyclic self-centralizing tori have order $(q^n \pm 1)/\gcd(2, q - 1)$ and automizer of order $2nf$, where $q = r^f$. But $(q^n \pm 1)/\gcd(2, q - 1) = (2n)^2 + 1$ only has the solutions given in cases (2) and (4). For A of type $D_n(q)$ with $n \geq 4$ the cyclic self-centralizing tori are of order $(q^n - 1)/\gcd(4, q^n - 1)$ with automizer of order n , and of order $q^{n-1} - 1$ with $q = 2$ with automizer of order $2(n - 1)$. These do not lead to examples. For groups of type ${}^2D_n(q)$ the cyclic self-centralizing tori are of order $(q^n + 1)/\gcd(2, q^n + 1)$ with automizer of order n , and of order $q^{n-1} + 1$ with $q = 2$ with automizer of order $2(n - 1)$. The only examples here are those in (2) and (4).

Now assume that $S = L_n(q)$ with $n \geq 2$. Here, cyclic self-centralizing tori have orders $(q^n - 1)/(q - 1)/d$ with automizer of order n , and $(q^{n-1} - 1)/d$ with automizer of order $n - 1$, where $d := \gcd(n, q - 1)$. This leads to $L_2(4) \cong \mathfrak{A}_5$, $L_2(9) \cong \mathfrak{A}_6$, $L_2(11)$, $L_3(4)$, $L_2(16).2$ and $L_2(256).8$. Finally, for unitary groups $S = U_n(q)$ with $n \geq 3$, cyclic self-centralizing tori have orders $(q^n - (-1)^n)/(q + 1)/d$ with automizer of order n , and $(q^{n-1} - (-1)^{n-1})/d$ with automizer of order $n - 1$, where $d := \gcd(n, q + 1)$. This gives $(A, p) = (U_3(11).2, 37)$ as the only example. \square

Finally we prove the last statement of the Introduction.

Proposition 6.3. *For any prime p with $\sqrt{p-1}$ an integer there are infinitely many finite solvable groups G with $|\text{Irr}_{p'}(G)| = 2\sqrt{p-1}$.*

Proof. By Dirichlet's theorem on arithmetic progressions there are infinitely many primes r of the form $pn + 1$ where n is an integer. Pick such an r and set $m := \sqrt{p-1}$. Let V be an m -dimensional vector space over the field with r elements. Then $\Gamma L(V)$ contains a subgroup $\Gamma L_1(r^m) \cong C_{r^m-1} \rtimes C_m$. Since p divides $r^m - 1$, this former group contains a (unique) subgroup A of the form $C_p \rtimes C_m$. We claim that $C_A(P) = P$ where P is the Sylow p -subgroup of A . Let x be a generator of P and let y be a generator of a cyclic subgroup of order m in A so that $x^y = x^r$. We have to show that whenever s is an integer with $1 \leq s < m$, then $x^{r^s} \neq x$. But this is clear since $r^m - 1$ does not divide $r^s - 1$.

Now set $G = V \rtimes A$. Then $O_{p'}(G) = F(G) = V$, VP is a Frobenius group, and $G/V = A$ is a Frobenius group of the form $C_p \rtimes C_m$. Now apply Lemma 6.1. \square

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